Department of Mathematics

Pattamundai College, Pattamundai

2nd Semester

Real Analysis
Core – 3

Sec–A
(Unit–1)

1. Closed interval is a closed set _____. (T/F)

2. Open interval is a open set _____. (T/F)

3. Write the difference between upper bound and least upper bound.

4. Given an example each of a bounded set which contains its (glb) but does not contain its (l.u.b)

5. Which of the following statements are true.

   i) $xy = 0 \Leftrightarrow x^2 + y^2 = 0$

   ii) $x^2 + y^2 = 0 \Leftrightarrow x^4 + y^4 = 0$

6. Find the supremum and infimum of the set

   S = { All rational numbers between $\sqrt{2}$ and $\sqrt{3}$ }

7. Let $I_n = (\frac{-1}{n}, 1+\frac{1}{n})$ be an open interval for $n \in N$, then find $\overset{\infty}{\underset{n=1}{\cup}} I_n$.

8. Write the definition of derived set.

9. $(A \cup B)' = A' \cup B'$ (T/F)

10. Interior of a set is a largest open set. Justify your answer.

11. What is the difference between the adherent point and limit point.

12. Let Sn = $[a+\frac{1}{n}, a+2]$, $n \in NI$, $a \in IR$ be a subset of R. Is $\overset{\infty}{\underset{n=1}{\cup}} S_n$ is a closed set?

13. Find the limit points of {1,2,3,4,5}

(Unit–2)

15. What is real sequence.

16. Define bounded and unbounded sequence.

17. What is null sequence and give a example of nuii-sequence.

18. If $x_n^2 \to L^2 \Rightarrow |x_n| \to |L|$ (T/F)

19. Suppose $\{x_n\}$ and $\{y_n\}$ are two convergent sequence then $\{x_n + y_n\}$ is convergent or divergent.

20. Give an example of two sequence which are divergent but their sum is conversent.

P.T.O.

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Or

$\{x_n\}$, $\{y_n\}$ are divergent sequence but $\{x_n+y_n\}$ is convergent sequence.

21. Give an example of Oscillates infinite sequence.

22. If $\lim\limits_{n\to\infty} x_n = l$ and $\lim\limits_{n\to\infty} y_n = m$ then

$$\lim\limits_{n\to\infty} \frac{x_1y_n + x_2y_{n+1} + \ldots_3\ldots\ldots x_ny_1}{n} = l.m \text{ (T/F)}$$

23. The sequence $<\frac{1}{n}>$ is a cauery sequence. (T/F)

24. The product of two cauchy sequence is a cauchy sequence. (T/F).

25. Every bounded sequence has a convergent subsequence. (T/F)

26. Every sequence has a monotonic subsequence. (T/F).

27. If S_1 and S_2 are positive real numbers and $S_{n+2} = \sqrt{S_{n+1}S_n}$

28. The value of $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ lies between.......

29. A sequence can converge to more than one limit.

30. Every bounded monotonic sequence is _____(T/F)

31. If $S_1 = \sqrt{2}$ and $S_{n+1} = \sqrt{2+\sqrt{sn}}$, $n\in Nl$, then find S_4.

32. Give the example of two sequence one is convergent and another is divergent then their sum is divergest.

33. Let f be a function on IR defined by

$$f(x) = \begin{cases} 1 & x \in Q \\ 1 & x \in Q^c \end{cases}$$

Show that f is discontinuous at every point of IR.

(Unit–3)

34. Show that the function f is defined on IR by

$$f(x) = \begin{cases} x & x \in Q \\ -x & x \in Q^c \end{cases}$$

is continuous only at x =0.

35. Show that the root of the equation $x^2-3 =0$ lies between [1,2]

36. Write the difference between the continuity and uniform continuity.

37. Every uniform continuos function is continuous. Justify for answer.

38. Given an example a function is continous of IR but not uniform continuous on IR.

39. The function $f(x) = \dfrac{x^2 - a^2}{x - a}$ at x = a is _____.

40. $\lim\limits_{x \to 0} f(x)$, where $f(x) = \dfrac{\sin[x]}{[x]}$

(Unit – 4)

41. Give an example of a function which is continuous every where but not differentiable at x= 0.

42. Continuity is a necessary or sufficient condition for derivative.

43. Give an example of a function which is no where differentiable.

44. Consider the function $f(x) = \begin{cases} 0 \text{ when } 0 \leq x \leq 1 \\ x+1 \text{ when } 1 < x \leq 2 \end{cases}$

    Show that f not satisfied the Rolles Theorem.

45. What is the difference between Rolle's and Lagrange mean value Theorem.

46. The function f(x) = |x+2| is not differentiable at a point

    a) x=2   (b) x =−2   (c) x=−1   (d) x=1

47. Give an example so that the function f(x) + g(x) is continuous in closed interval [a,b] but not differentiable.

48. If f defined and derivable on [a,b] with f(a) =f(b) =0 and f′(a)f′(b)> 0. then f must varish at least one in (a,b) (T/F).

49. If f(x+y) = f(x) f(y) for all x and y. Suppose that f(3) = 3 and f′(0) =11, Then f′(3) is equal to _____?

50. If f′(a) exists, then $\lim\limits_{n \to 0} \dfrac{f(a+h) - f(a-h)}{h}$ is equal to _____?

Sec – B   [2.5
(Unit–1)

1. Show that there is no rational number whose square is 2.

2. The set N of natural numbers is not bounded above. prove it.

3. Prove that for all real numbers x and y.

    |x+y|² + |x−y|² = 2 |x|² + 2|y|²

4. Use Archimedian property of real numbers, show that if $a,b \in R$ such that $a \leq b + \frac{1}{n}$ ∀ $n \in N$ then a≤b.

5. Suppose A is any set then prove that A′, A′′...... A$^{(n)}$ are closed set where A′ is the derived set of A. A is the inderived set of A′ and A$^{(n)}$ the n_{th}x derived set of A.

6. Prove that the intersection of two open sets is open.

7. The intersection of two closed sets is a closed set.

8. A set S is closed $\Leftrightarrow$ $\overline{S}$ = S.

9. Let A,B $\subseteq$ R, Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$

10. Show that the finite set or a countable set is a closed set.

P.T.O.

(Unit–2)

11. Find all the limit points of the sequence

$$<x_n> = \{1,0,0,4,0,6,0,8,9,10,0 ....\}$$

Or

$$<x_n> \begin{cases} 0 \text{ if } n \text{ is prime} \\ n \text{ if } n \text{ is not prime} \end{cases}$$

12. Find all the limit points of $<x_n> = \sin \frac{1}{2} n\pi$, $n \in z$.

13. Prove that a bounded sequence with a unique limit point is convergent.

14. Show that $\displaystyle\lim_{x \to \infty} \frac{1+2+3+4+5........+n}{n} = \frac{1}{2}$

15. If $\displaystyle\lim_{x \to \infty} \{x_n\} = \ell \Rightarrow \lim_{x \to \infty} |x_n| = |\ell|$ but the converse is not true.

16. Show that $\displaystyle\lim_{x \to \infty} \frac{1}{n} (1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+....+n)^{1/n} = 1$.

17. Show that $<x_n>$, where $<x_n> = \dfrac{(-1)^n}{n}$ is a convergent sequence.

18. Show that a monotonically increasing sequence either converges to its supremum or diverges to $+\infty$.

19. A bounded sequence $<x_n>$ convergent to l if and only if $\displaystyle\lim_{x \to \infty} \sup <x_n> = \lim_{x \to \infty} \text{Inf } f(x_n) f<x_n> = l$.

20. If $<x_n>$ and $<y_n>$ are bounded sequence such that $x_n \le y_n \; \forall n \in N$ then show that $\displaystyle\lim_{x \to \infty} \sup(x_n) \le \lim_{x \to \infty} \sup(y_n)$.

21. Prove that $\{x_n\} \le \{y_n\}$ then

   i) $\{y_n\}$ is convergent $\Rightarrow \{x_n\}$ is convergent

   ii) $\{x_n\}$ is divergent $\Rightarrow \{x_n\}$ is convergent

22. Find the limit superior and limit in ferior of the sequence $x_n = (-1)^n+1$, $n \in N$

23. Find the limit superior and limit Inferior of the sequence $\{x_n\} = \sin\left(\dfrac{n\pi}{2}\right)$, $n \in N$

(Unit–3)

24. Determine the constnat a and b so that the function f defined below is continuous every where

$$f(x) = \begin{cases} 2x+1 & \text{if } x \le 1 \\ ax^2+b & \text{if } 1<x<3 \\ 5x+2a & \text{f } x \ge 3 \end{cases}$$

25. Show that the function is continuos at $x = 0$

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

26. $\displaystyle\lim_{x \to 2}(2x+3) = 7$ by the method of epsilon-delta approach.

27. A function f defined on R is continuous on R if andonly if for each open set G in R then $f^{-1}(G)$ is open in R.

28. If a function f is continuous on a closed interval [a,b] then it is uniform continuous on [a,b].

29. Is the function $f(x) = \frac{x}{x+1}$ uniformly continuous for $x \in [0,2]$. Justify your answer.

30. Let $f(x) = x^2$, $x \in R$. show that f is unifermly continous on every closed and finite interval but is not uniformly continuos on R.

31. Show that sin x is uniformly continous on R.

32. Suppose f(x) and g(x) are two uniformly continous function then f(x) g(x) is continous function?

33. Show that the function f defined by

$$f(x)_z \begin{cases} 1 \text{ if x is rational} \\ 0 \text{ if x is irrational} \end{cases}$$

is discontinuous at every point.

34. Show that $f(x) = \lim_{x \to \infty} \frac{x^{2n}}{1+x^{2n}}$ is continous at all points of R except $x = \pm 1$.

35. Let f, g be two functions defiend as follows

$f(x) = \sqrt{x}$, $x \geq 0$, $g(x) = x^2 + 1$ ∀ $x \in R$ show that $(fog)(x) = \sqrt{x^2+1}$

(Unit–4)

36. If a function is differentiable in a closed interval [a,b] and $f^1(a)$, $f^1(b)$ are opposite signs then ∃ atleast one point C of the open interval (a,b) such that $f(c)' = 0$ prove it.

37. If the derivative of f is exist in [a,b] and $f'(a) \neq f'(b)$ and K is any number lying between f(a) and f(b) then ∃ atleast one point C (a,b) such that $f'(c) = k$.

38. Verify whether the function $f(x) = \sin x$ in $[0,\pi]$ satisfies the conditions of Rolle's theorem and hence find c.

39. Use the mean value theorem prove that $\frac{x}{1+x^2} < \tan^{-1}x < x$ if x>0

40. Evaluate the value of θ. That appers in Lagrange mean value theoremfor the function $x^2 - 2x + 3$ given that a =1 and b = $\frac{1}{2}$.

41. Vefify Lagrange mean value theorem for the function $f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

42. Evaluate the value of Q that appears in Lagranges mean value theorem for the function $x^2 - 2x + 3$ given that a =1 and b = $\frac{1}{2}$.

43. Separate the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x - 7$, for all $x \in R$ in increasing or decreasing

44. Sow that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$, x>0.

P.T.O.

45. $\dfrac{x^2}{2} < x - \log(1+x) < \dfrac{x^2}{2(x+1)}$, $-1 < x < 0$

46. Prove that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ where $0 < x < \dfrac{\pi}{2}$

47. Write the statement of cauchy's mean value theorem.

48. Find the difference between Cauchy mean value theorem and Generallised mean vale theorem.

49. What is mean value theorem. (Write the statement only)

50. Show that $\dfrac{Sin\alpha - Sin\beta}{Cos\beta - Cos\alpha} = Cot\theta$ where $0 < \alpha < \theta < \beta <$

Sec – C   [5 Mark

(Unit–1)

1. State and prove the Archimedean Property of R.

2. A point P is a limit point of a set S if and only if every neighbourhood of p contains infinite many point of S.

3. If a set S is bounded then it has lub iff it has g.l.b. prove it.

4. Let $T = \{\dfrac{1}{n}, n \in N\} \cup \{1 + \dfrac{3}{2n}, n \in N\}$ $\{6 - \dfrac{1}{3}, n \in N\}$. Find derived set T' of T. Also find superemum of T and greatest number of T.

(Unit–2)

5. State and prove Bolzano- weierstrass theorem for sequences.

Or

Every bonded sequence has a limit point

6. Show that $\lim\limits_{n\to\infty} r^n = 0$, if $|r| < 1$.

7. Show that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

8. If $\lim\limits_{n\to\infty} fn = l$ then $\lim\limits_{n\to\infty} \dfrac{f1 + f2 + f3 + ... + fn}{n} = 1$

9. If $\{xn\}$ is a sequence such that $\lim\limits_{n\to\infty} \left(\dfrac{x_{n+1}}{x_n} \right) = l$

   i) When $|l| < 1$ then $\lim\limits_{n\to\infty} \{xn\} = 0$

   ii) When $|l| > 1$ then $\lim\limits_{n\to\infty} \{xn\} = \infty$

10. If $<fn>$ is a sequence of positive numbers such that

    $fn = \frac{1}{2}(f_{n-1} + f_{n-2})$ for all $n \geq 3$), then show that $<fn>$ converges to $\left(\dfrac{f1 + 2f2}{3} \right)$

11. Let $<an>$ be a sequence defined as $a_1 = \frac{3}{2}$, $a_{n+1} = 2 - \dfrac{1}{an}$, $n \geq 1$, Show that $\langle a_n \rangle$ is monotonic and bounded and also converges to 1.

(Unit–3)

12. A function f defined on an interval I is continuous at $a \in I$ if and only if for every sequence <an> in I which converses to a, then $\lim_{n \to \infty}$ f(an) = f(a)

13. Let f be a function defined on (0,1) by

$$f(x) \begin{cases} 0 \text{ if } x \in Q^C. \\ \dfrac{1}{q} \text{ if } x = \dfrac{p}{q}, \text{ g.c.d } (p,q)=1, x \in Q \end{cases}$$

   Prove that f is continuous at each irrational point at discontinuous at each rational point.

14. If a function f is continous in a closed bounded interval [a,b], then it is bounded.

15. If a function f is continuous in an interval [a,b] and f(a) $\neq$ f(b), then f assumes every value between f(a) and f(b)

16. If a continuous function of x satisfies the functional equation f(x+y) = f(x) + f(y) then show that f(x)=αx where α is a constant.

(Unit– 4)

17. The function f defined by

$$f(x) = \begin{cases} x^2+3x+a & \text{if } x \leq 1 \\ b+2 & \text{if } x>1 \end{cases}$$

   is given to be differentiable for every x. Find the value of a and b.

18. Show that the function

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   is continous at x =0 but not differentiable at x =0.

Or

   In general, $f(x) = \begin{cases} x^{2n+1} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x=0. \end{cases}$ prove that f^n exists for all $x \in R$, f^n is continuos at x = 0 but f^n is not differentiable at x =0.

19. State and prove Rolle's theorem.

20. State and prove the Lagrange's mean value theorem.

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